

ANALYSIS OF EIGENVALUE BOUNDS FOR REAL SYMMETRIC INTERVAL MATRICES

**FANG SU¹, ZHAN XU², LIZHI CHENG¹
and SHENGGUO LI¹**

¹Department of Mathematics
National University of Defense Technology
Changsha 410073
P. R. China
e-mail: sufang@lsec.cc.ac.cn

²School of Electronic Science and Engineering
National University of Defense Technology
Changsha 410073
P. R. China

Abstract

In this paper, we present several verifiable conditions for eigenvalue intervals of real symmetric interval matrices overlapping or not overlapping. To above cases, two new methods with algorithms for computing eigenvalue bounds of real symmetric matrices are developed. We can estimate eigenvalue bounds moving away the assumption that two intervals containing two eigenvalues of real symmetric interval matrices are not overlapping. These methods can analyse stability of systems in control fields extensively. Numerical examples illustrating the applicability and effectiveness of the new methods are also provided.

2010 Mathematics Subject Classification: 65G10.

Keywords and phrases: interval matrix, eigenvalue bounds, real symmetric matrices, eigenvalue interval overlapping, Weyl's theorem.

Received May 11, 2010

© 2010 Scientific Advances Publishers

1. Introduction

Many real-life problems suffer from diverse uncertainties, as a result of inaccuracy of measurements, errors in manufacture, etc. Therefore, the concept of uncertainty is becoming more and more important. Probability theory is the traditional approach to handling uncertainty. This approach requires sufficient statistical data to justify the assumed statistical distributions. Analysis agrees that, given sufficient statistical data, the probability theory describes the stochastic uncertainty well. However, probabilistic modelling cannot handle situations with incomplete or little information on which to evaluate a probability, or when that information is nonspecific, ambiguous, or conflicting. In the mid sixties, the interval analysis was proposed [5]. It turns out to be a very powerful technique to study the variations of a system and to understand its uncertainty. One of the most important properties of this approach is the fact that, it is possible to certify the results of all the states of a system.

The problem of computing the eigenvalue bounds of interval matrices has been studied since the nineties. Deif [1] firstly considered the interval eigenvalue problem, and gave the range of eigenvalues of an interval matrix [9]. They proposed the exact bounds under certain assumptions on the sign pattern invariancy of the corresponding eigenvectors, such assumptions are not easy to verify [2]. Qiu [6] presented some theoretical results based on an interval perturbation approximating formula. However, it can only produce approximate, but not true bounds for eigenvalues. Qiu [7] dealt with the standard interval eigenvalue problem by using the vertex solution theorem and the parameter decomposition solution theorem. Rohn [10] proved several theorems for the real eigenvalues under three assumptions. Zhan [11] presented the range of the smallest and largest eigenvalues of real symmetric interval matrices. Leng [3] obtained the eigenvalue bounds of the original interval eigenvalue problem based on the matrix perturbation. This method is very simple and unconditional, but the bounds are not very sharp. Leng [4] presented a new method with two algorithms for computing bounds to real eigenvalues of real-interval matrices. But, they supposed that two intervals containing two eigenvalues of the interval eigenvalue problem are not overlapping.

In this paper, we firstly give several simple verifiable conditions justifying eigenvalue intervals of real symmetric interval matrices overlapping or not overlapping. Then, two new methods with algorithms for computing eigenvalue bounds of real symmetric matrices are showed. When eigenvalue intervals occur overlapping, the eigenvalue bounds can also be obtained by these methods, which other methods have the additional conditions.

2. Bounds of Interval Eigenvalues

Consider the real symmetric interval eigenvalue problem

$$Au = \lambda u. \quad (1)$$

Here $A \in A^I$, interval matrix A^I is defined as

$$A^I = [\underline{A}, \overline{A}] = \{A \in R^{n \times n}; \underline{A} \leq A \leq \overline{A}\}, \quad (2)$$

where $\underline{A}, \overline{A} \in R^{n \times n}; \underline{A} \leq \overline{A}$, are given symmetric matrices. It is important to note that not every matrix in A^I is symmetric. Here, we only consider the symmetric matrices, and the non-symmetric parts are considered in our future paper. By

$$\begin{aligned} A^c &= \frac{\overline{A} + \underline{A}}{2} \quad \text{for} \quad a_{ij}^c = \frac{\overline{a_{ij}} + \underline{a_{ij}}}{2}, \\ \Delta A &= \frac{\overline{A} - \underline{A}}{2} \quad \text{for} \quad \Delta a_{ij} = \frac{\overline{a_{ij}} - \underline{a_{ij}}}{2}, \end{aligned}$$

we denote the midpoint and the radius of A^I , respectively.

λ is the eigenvalue of the uncertain-but-bounded matrix A and u is the corresponding eigenvector of λ . For a given real symmetric interval matrix A^I , find an eigenvalue interval λ^I defined by

$$\lambda^I = [\underline{\lambda}, \overline{\lambda}] = (\lambda_i^I), \quad \lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i],$$

such that it encloses all possible eigenvalues λ satisfying $Au = \lambda u$, where $A \in A^I$, and also it should be as small as possible.

Since the eigenvalues are now not only the points, but also the intervals. There may be occurring overlapping of eigenvalue intervals. We give some justified conditions are based on the Weyl's theorem [12], at first, we review some knowledge about the Weyl's theorem.

Theorem 2.1 (Weyl 1912). *Let $A, B \in R^{n \times n}$ be symmetric matrices with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq \dots \geq \lambda_n(B)$, respectively. Then one has*

$$\lambda_i(A + B) \geq \begin{cases} \lambda_i(A) + \lambda_n(B); \\ \lambda_{i+1}(A) + \lambda_{n-1}(B); \\ \dots \\ \lambda_n(A) + \lambda_i(B); \end{cases} \quad (3)$$

and

$$\lambda_i(A + B) \leq \begin{cases} \lambda_i(A) + \lambda_1(B); \\ \lambda_{i-1}(A) + \lambda_2(B); \\ \dots \\ \lambda_1(A) + \lambda_i(B). \end{cases} \quad (4)$$

Theorem 2.2. *Let $A \in R^{n \times n}$ be symmetric matrices in $A^I, A^c, \Delta A$ are the midpoint and the radius of A^I , respectively. The eigenvalues of above matrices hold that*

$$\lambda_i(A^c) - \rho(\Delta A) \leq \lambda_i(A) \leq \lambda_i(A^c) + \rho(\Delta A). \quad (5)$$

Proof. For $\delta A \in [-\Delta A, \Delta A]$, one has $A = A^c + \delta A$, by Weyl's Theorem 2.1,

$$\lambda_i(A^c) + \lambda_n(\delta A) \leq \lambda_i(A) \leq \lambda_i(A^c) + \lambda_1(\delta A), \quad \forall i = 1, \dots, n.$$

According to spectral radius definition, we have

$$\lambda_1(\delta A) \leq \rho(\delta A), \quad \lambda_n(\delta A) \geq -\rho(\delta A),$$

whence

$$\lambda_i(A^c) - \rho(\delta A) \leq \lambda_i(A) \leq \lambda_i(A^c) + \rho(\delta A).$$

As $\delta A \in [-\Delta A, \Delta A]$, we get $\rho(\delta A) \leq \rho(\Delta A)$, so,

$$\lambda_i(A^c) - \rho(\Delta A) \leq \lambda_i(A) \leq \lambda_i(A^c) + \rho(\Delta A).$$

Corollary 2.1. *Let $A \in R^{n \times n}$ be symmetric matrices in A^I , A^c , ΔA are the midpoint and the radius of A^I , respectively. If the eigenvalues of A^c and ΔA satisfy*

$$\lambda_{i+1}(A^c) + \rho(\Delta A) < \lambda_i(A^c) - \rho(\Delta A), \forall i = 1, 2, \dots, n-1,$$

then the eigenvalue intervals of eigenvalue problem (1) are separating.

Proof. From Theorem 2.2, the lower bound of i -th eigenvalue exceeds the upper bound of $(i+1)$ -th eigenvalue, they are obviously not overlapping.

Remarks 2.1. If characteristic polynomial of A has the repeated roots, eigenvalue intervals are obviously overlapping.

3. Algorithms for Computing the Eigenvalue Bounds

Let $A^I = [\underline{A}, \overline{A}]$ is a real symmetric interval matrix, A^c be the midpoint of A^I , and its eigenvalues be $\lambda_1(A^c) \geq \lambda_2(A^c) \geq \dots \geq \lambda_n(A^c)$. Let the characteristic polynomial of $A \in A^I$ be denoted by $P(\lambda)$, which is a continuous function of variables λ . Algorithm 3.1 is based on the ideals in [4], and the algorithm provides the tighter upper bounds to all eigenvalues for a given precision ϵ . Here, we choose initial outer upper bound according to Theorem 2.2, it will converge faster. The computation of the lower bound to all eigenvalues makes the same way.

Algorithm 3.1. (Characteristic polynomial algorithm)

1. compute eigenvalues $\lambda_1(A^c) \geq \lambda_2(A^c) \geq \dots \geq \lambda_n(A^c)$;
2. compute eigenvalues $\lambda_1(\Delta A) \geq \lambda_2(\Delta A) \geq \dots \geq \lambda_n(\Delta A)$;

3. determine spectra radius $\rho(\Delta A) = \max_{i=1, \dots, n} (|\lambda_i(\Delta A)|)$
4. define $c = [\lambda_1(A^c) + \rho(\Delta A), \lambda_2(A^c) + \rho(\Delta A), \dots, \lambda_n(A^c) + \rho(\Delta A)]$;
5. **for** $i = 1, \dots, n$
6. Initialize $lb(i) = (\lambda_i(A^c) + c(i)) / 2$;
7. Initialize $ub(i) = c(i)$;
8. **while** $((ub(1) - lb(1)) \geq \epsilon)$
9. **If** $\min_{lb(i) \leq \lambda \leq ub(i)} P(\lambda_i) \cdot \max_{lb(i) \leq \lambda \leq ub(i)} P(\lambda_i) > 0$;
10. $ub(i) = lb(i)$, $lb(i) = ub(i) - (ub(i) - \lambda_i(A^c)) / 2$;
11. **else**
12. $lb(i) = lb(i) + (ub(i) - lb(i)) / 2$;
13. **end**
14. **end**
15. **return** ub

Remark 3.1. In step 7, we consider $c(i) = \lambda_i(A^c) + \rho(\Delta A)$ $i = 1, 2, \dots, n$ as the initial outer upper bounds for eigenvalue intervals. In step 9, according to whether the sign of characteristic polynomial $P(\lambda)$ remains unchanged in a given interval, the iterative $[lb, ub]$ is gradually reduced until its length is less than a given precision ϵ . This theory was presented by Rohn in [8].

If the radius of A^I is not small, it occurs eigenvalue intervals overlapping in all probability. When the eigenvalue intervals occurring overlapping, we can not using the sign pattern invariance of characteristic polynomial. We can estimate the eigenvalue bounds according to Weyl Theorems 2.1 and 2.2. Algorithm 3.2 is based on Weyl theorem.

Algorithm 3.2. (Weyl algorithm)

1. compute eigenvalues $\lambda_1(A^c) \geq \lambda_2(A^c) \geq \dots \geq \lambda_n(A^c)$;
2. compute eigenvalues $\lambda_1(\delta A) \geq \lambda_2(\delta A) \geq \dots \geq \lambda_n(\delta A)$;
3. **for** $i = 1, \dots, n$
4. $ub(i) = \max_{\delta A \in \Delta A} \min_{k=1, \dots, i} \{\lambda_k(A^c) + \lambda_{i-k+1}(\delta A)\}$
5. $lb(i) = \min_{\delta A \in \Delta A} \max_{k=i, \dots, n} \{\lambda_k(A^c) + \lambda_{i-k+n}(\delta A)\}$
6. **end**
7. **return** $ub, lb, i = 1, \dots, n$

Remark 3.2. In step 4, we determine upper bound of eigenvalue λ_i according to (4), that is, we select the smaller upper bound. The same reason is for lower bound of eigenvalue in step 5.

4. Numerical Result

Example 4.1. (A spring-mass system with four degrees of freedom [3]).

We consider a spring-mass system with four degrees of freedom as shown in Figure 1. Masses are denoted by m_1, m_2, m_3, m_4 , and springs are denoted by k_1, k_2, k_3, k_4 , and k_5 , respectively.

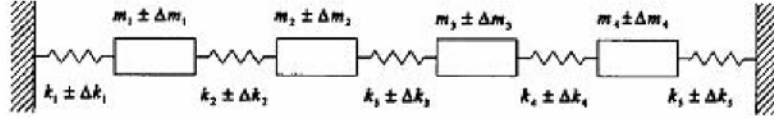


Figure 1. A spring-mass system with four degrees of freedom.

In order to take the form shown in problem (1), let the nominal mass matrix M^c and the deviation radius matrix of the mass matrix simplified be given, respectively, by

$$M^c = \text{diag}(1, 1, 1, 1),$$

and

$$\Delta M = \text{diag}(0, 0, 0, 0).$$

The nominal stiffness matrix K^c and the deviation radius matrix of the stiffness matrix are given, respectively, by

$$K^c = \begin{bmatrix} 3000 & -2000 & 0 & 0 \\ -2000 & 5000 & -3000 & 0 \\ 0 & -3000 & 7000 & -4000 \\ 0 & 0 & -4000 & 9000 \end{bmatrix}, \quad (6)$$

and

$$\Delta K = \begin{bmatrix} 25 & 15 & 0 & 0 \\ 15 & 35 & 20 & 0 \\ 0 & 20 & 45 & 25 \\ 0 & 0 & 25 & 55 \end{bmatrix}. \quad (7)$$

Let the upper and lower eigenvalue bounds by using Algorithm 3.2 be denoted by $\overline{\lambda}_i$ and $\underline{\lambda}_i$, those in [3] be denoted by $\overline{\mu}_i$ and $\underline{\mu}_i$, for $i = 1, 2, 3, 4$, respectively. The results are summarized in Table 1. It shows that, the present method can produce the tighter eigenvalue bounds as the method in [3] does. The results by Leng in [4], show good agreement with Algorithm 3.1. However, in [4] can only deal with the case that the eigenvalue intervals do not overlap. Here, we emphasize the Weyl's theorem, which it can obtain all the eigenvalue bounds of symmetric matrix. The results show that Algorithm 3.2 can provide satisfied eigenvalue bounds.

Table 1. Interval eigenvalues by Algorithm 3.2 and the method in [3]

| | $\underline{\lambda}_i$ | $\overline{\lambda}_i$ | $\underline{\mu}_i$ | $\overline{\mu}_i$ |
|---------------|-------------------------|------------------------|---------------------|--------------------|
| λ_1^I | 825.25974 | 985.06321 | 815.16148 | 995.16148 |
| λ_2^I | 3309.94664 | 3469.75011 | 3299.84838 | 3479.84838 |
| λ_3^I | 6984.55708 | 7144.36055 | 6974.45882 | 7154.45882 |
| λ_4^I | 12560.62959 | 12720.43306 | 12550.53133 | 12730.53133 |

Example 4.2. Consider the symmetric interval matrix

$$A^c = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 6 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad (8)$$

and

$$\Delta A = \begin{bmatrix} 1 & 5 & 2 \\ 5 & 2 & 4 \\ 2 & 4 & 2 \end{bmatrix}. \quad (9)$$

Let the upper and lower eigenvalue bounds by using Algorithm 3.2 be denoted by $\overline{\lambda}_i$ and $\underline{\lambda}_i$, those by using Theorem 2.2 be denoted by $\overline{\mu}_i$ and $\underline{\mu}_i$, for $i = 1, 2, 3$, respectively. The results are summarized in Table 2.

Table 2. Interval eigenvalues by Algorithm 3.2 and using Theorem 2.2

| | $\underline{\lambda}_i$ | $\overline{\lambda}_i$ | $\underline{\mu}_i$ | $\overline{\mu}_i$ |
|---------------|-------------------------|------------------------|---------------------|--------------------|
| λ_1^I | 2.5329 | 16.0881 | -2.2298 | 16.0881 |
| λ_2^I | -0.0252 | 7.2107 | -6.3445 | 11.9734 |
| λ_3^I | -8.9026 | 3.0961 | -8.9026 | 9.4154 |

Because the eigenvalue intervals are overlapping, we cannot use Deif's method, Leng's method, etc. It shows that Algorithm 3.2 can provide more accurate eigenvalue bounds.

5. Conclusion

Based on Weyl's Theorem 2.1, we give some verifiable conditions for eigenvalue intervals overlapping or not overlapping. We can obtain credible eigenvalue bounds despite some large according to Theorem 2.2. The bounds can be good initial bounds for Algorithm 3.1 or for the algorithm in [4]. Algorithm 3.2 for estimating eigenvalue bounds of real symmetric interval matrix is presented, which can deal with cases whether overlapping or not overlapping. Numerical examples show that our methods with algorithms are effective and reliable.

Acknowledgement

Authors of this paper thank The State Key Laboratory of Science-Engineering Computing (China) for their great support. This work was supported by National Natural Science Foundation of China (No. 60573027) and Science Research Project of National University of Defense Technology (No. JC09-02-05). Fang Su would like to acknowledge the support of Department of Mathematics and Systems Science, National University of Defense Technology.

References

- [1] A. Deif, The interval eigenvalue problem, *Z. Angew. Math. Mech.* 71(1) (1991), 61-64.
- [2] A. Deif and J. Rohn, On the invariance of the sign pattern of matrix eigenvectors under perturbation, *Linear Alge. Appl.* 196 (1994), 63-70.
- [3] H. N. Leng and Z. Q. He, Computing eigenvalue bounds of structures with uncertain-but-non-random parameters by a method based on perturbation theory, *Comm. Num. Meth. Engg.* 23 (2007), 973-982.
- [4] H. N. Leng, Z. Q. He and Q. Yuan, Computing bounds to real eigenvalues of real-interval matrices, *Int. J. Numer. Meth. Engg.* 74 (2008), 523-530.
- [5] R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, New York, 1966.
- [6] Z. P. Qiu, S. H. Chen and I. Elishakoff, Bounds of eigenvalues for structures with an interval description of uncertain-but-non-random parameters, *Chaos Solitons Fractals* 7 (1996), 425-434.
- [7] Z. P. Qiu and X. J. Wang, Solution theorems for the standard eigenvalue problem of structures with uncertain-but-bounded parameters, *J. Sound Vib.* 282 (2005), 381-399.

- [8] J. Rohn, Stability of interval matrices: The real eigenvalue case, *IEEE Trans. Auto. Contr.* 37(10) (1992), 1604-1605.
- [9] J. Rohn and A. Deif, On the range of eigenvalues of an interval matrix, *Computing* 47(3-4) (1992), 373-377.
- [10] J. Rohn, Interval matrices: Singularity and real eigenvalues, *SIAM J. Matrix Anal. Appl.* 14(1) (1993), 82-91.
- [11] X. Z. Zhan, Extremal eigenvalues of real symmetric matrices with entries in an interval, *SIAM J. Matrix Anal. Appl.* 27(3) (2006), 851-860.
- [12] X. D. Zhang, *Matrix Analysis and Applications*, Tsinghua University Press, Springer, 2005.

